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Quantum dynamics and decoherence in generalized Coleman-Hepp model and boson detector model

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Abstract. Decoherence process in quantum mechanical entangled spin states is formulated and solved on the basis of a generalized Coleman-Hepp model and a boson detector model. These models are exactly solved to give reduced density matrices and the von Neumann entropy. Detailed studies are done on environmental fluctuations which cause decoherence in the correlated entangled states: a single detector model and two detectors model are examined with the use of analytic solutions and numerical evaluations.

PACS. 03.65.-w Quantum mechanics – 05.30.-d Quantum statistical mechanics

1 Introduction

In a series of papers [1–4], we studied decoherence process of quantum systems. This was performed on the basis of a generalized version [5,6] of Coleman-Hepp (CH) model [7] and boson detector (BD) model [8]. Our main interest lies in spin degree of freedom of an incident particle into a detector which is composed of an array of spins (CH) or harmonic oscillators (BD). Degree of decoherence manifests itself in expectation values of certain observables, quasi-probability density and von Neumann entropy. Among others, the degree of decoherence is clearly seen when dynamical time evolution of the von Neumann entropy $\mathcal{S}$ is determined:

$$\mathcal{S} \equiv -k_{\mathrm{B}} \mathrm{Tr} \rho \ln \rho \tag{1}$$

where ρ is the density matrix, k_{B} being the Boltzmann constant.

Once an eigenvalue problem,

$$\rho|\lambda_j\rangle = \lambda_j|\lambda_j\rangle, \tag{2}$$

is solved, we find

$$\mathcal{S} = -k_{\mathrm{B}} \sum_j \lambda_j \ln \lambda_j. \tag{3}$$

In our previous work [4,9], we were able to solve (2) even for the detector as well as for the incoming particle enabling us to calculate $\mathcal{S}$ exactly and found a *reciprocity relation of entropy*.

Thus we could solve the dynamical decoherence processes of CH and the BD models completely. In this paper, we further develop the previous theories to include entangled states which were recognized to be very important as early as 1935 by Schrödinger [10] and EPR (Einstein-Podolsky-Rosen) [11], and later by Bell [12]. Moreover, the quantum entanglement has played an important role in the field of quantum information [13]. The above mentioned extension will be done in the following sections.

2 Entanglement in Coleman-Hepp model

This section treats the generalized CH model [1,2,5,6] to make detailed studies on decoherence process of an entangled state.

2.1 Single detector model

Let us extend CH model to include two particles named A and B and a detector. Each of the two particles has a spin of magnitude 1/2 and the two spins are initially assumed to be in an entangled state. One of the two particles (A) moves toward the positive direction of x-axis and the other (B) stays always at the origin while the detector composed of N spins is placed in a one-dimensional array. The lth spin in the detector is located at the position x_l ($l = 1, \cdots, N$) with an interval d, $x_{l+1} - x_l \equiv d$. The particle A moves with a velocity v_{A} and interacts with the spins at each site of the detector.

The Hamiltonian of this model is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{P}_+^{\mathrm{A}} \mathcal{H}_1^{\mathrm{A}} \tag{4}$$

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with the projection operator $\mathcal{P}_{\pm}^{\mathrm{A}} = 1/2 \pm I_{\mathrm{A}}^z$, where I_{A}^z is the z-component of A's spin operator $\mathbf{I}_{\mathrm{A}}$ and similarly we define $\mathbf{I}_{\mathrm{B}}$.

The first term of equation (4) is the free Hamiltonian,

$$\mathcal{H}_0 = \mathcal{H}_{\mathrm{S}} + \mathcal{H}_{\mathrm{D}}, \tag{5}$$

$$\mathcal{H}_{\mathrm{S}} = v_{\mathrm{A}} P_{\mathrm{A}} + \hbar\omega_{\mathrm{A}} I_{\mathrm{A}}^z + \hbar\omega_{\mathrm{B}} I_{\mathrm{B}}^z, \tag{6}$$

$$\mathcal{H}_{\mathrm{D}} = \sum_{l=1}^{N} \hbar\omega_l S_l^z \tag{7}$$

where $\mathcal{H}_{\mathrm{S}}$ is the Hamiltonian for the particles A and B and $\mathcal{H}_{\mathrm{D}}$ for the detector. In equation (6) P_{A} is the momentum operator for the particle A and ω_{A}, ω_{B} are the angular frequencies of the spins of A and B: $\mathbf{S}_l = (S_l^x, S_l^y, S_l^z)$ is the lth spin operator of the detector. In equation (4), the interaction Hamiltonian between the particle A and the detector is given by

$$\mathcal{H}_1^{\mathrm{A}} = \frac{1}{2}\sum_{l=1}^{N} \hbar\Omega_l(X_{\mathrm{A}} - x_l) \times \left(\mathrm{e}^{\mathrm{i}\omega_l X_{\mathrm{A}}/v_{\mathrm{A}}} S_l^- + \mathrm{e}^{-\mathrm{i}\omega_l X_{\mathrm{A}}/v_{\mathrm{A}}} S_l^+\right) \tag{8}$$

where $\Omega_l(x)$ represents the interaction strength, X_{A} being the position operator of the particle A with further definition of $S_l^{\pm} = S_l^x \pm \mathrm{i} S_l^y$.

Using the total Hamiltonian, we can rewrite time evolution operator in a useful form,

$$\mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar} = \mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar}(\mathcal{P}_+^{\mathrm{A}} + \mathcal{P}_-^{\mathrm{A}}) \tag{9}$$

$$= \mathcal{D}_{\mathrm{A}}^z(\{\omega_l X_{\mathrm{A}}/v_{\mathrm{A}}\})\{\mathrm{e}^{-\mathrm{i}(\mathcal{H}'_0+\mathcal{H}'^{\mathrm{A}}_1)t/\hbar}\mathcal{P}_+^{\mathrm{A}} + \mathrm{e}^{-\mathrm{i}\mathcal{H}'_0 t/\hbar}\mathcal{P}_-^{\mathrm{A}}\}\mathcal{D}_{\mathrm{A}}^z(\{\omega_l X_{\mathrm{A}}/v_{\mathrm{A}}\})^\dagger \tag{10}$$

$$= \mathcal{D}_{\mathrm{A}}^z(\{\omega_l X_{\mathrm{A}}/v_{\mathrm{A}}\})\mathrm{e}^{-\mathrm{i}\mathcal{H}'_0 t/\hbar} \times\{V_{\mathrm{A}}(t)\mathcal{P}_+^{\mathrm{A}} + \mathcal{P}_-^{\mathrm{A}}\}\mathcal{D}_{\mathrm{A}}^z(\{\omega_l X_{\mathrm{A}}/v_{\mathrm{A}}\})^\dagger \tag{11}$$

where

$$\mathcal{D}_{\mathrm{A}}^z(\{\phi_l\}) = \prod_{l=1}^{N} \mathcal{D}_l^z(\phi_l) \tag{12}$$

$$\mathcal{D}_l^z(\phi_l) = \mathrm{e}^{-\mathrm{i}\phi_l S_l^z} \tag{13}$$

and

$$\mathcal{H}'^{\mathrm{A}}_1 = \sum_{l=1}^{N} \hbar\Omega_l(X_{\mathrm{A}} - x_l) S_l^x, \tag{14}$$

$$V_{\mathrm{A}}(t) = \exp\left[-\mathrm{i}\sum_{l=1}^{N} \Theta_l^{\mathrm{A}}(X_{\mathrm{A}}; t) S_l^x\right], \tag{15}$$

$$\Theta_l^{\mathrm{A}}(x; t) = \int_0^t dt' \Omega_l(x + v_{\mathrm{A}} t' - x_l), \tag{16}$$

with $\mathcal{H}'_0 = \mathcal{H}_{\mathrm{S}}$ and $\mathcal{P}_+^{\mathrm{A}} + \mathcal{P}_-^{\mathrm{A}} = 1$.

We note that the interaction effect is contained only in $V_{\mathrm{A}}(t)$ term of (11).

An initial density matrix is assumed to be

$$W(0) = |I\rangle\langle I| \otimes |\Psi\rangle\langle\Psi| \otimes \prod_{l=1}^{N} |\mathbf{z}_l^0\rangle\langle\mathbf{z}_l^0|, \tag{17}$$

where the spins of the two particles are in a singlet entangled state:

$$|I\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_{\mathrm{A}} \otimes |-\rangle_{\mathrm{B}} - |-\rangle_{\mathrm{A}} \otimes |+\rangle_{\mathrm{B}}\right) \tag{18}$$

with $I_{\mathrm{A}}^z|\pm\rangle_{\mathrm{A}} = \pm\frac{1}{2}|\pm\rangle_{\mathrm{A}}$ and $I_{\mathrm{B}}^z|\pm\rangle_{\mathrm{B}} = \pm\frac{1}{2}|\pm\rangle_{\mathrm{B}}$.

The orbital state of the particles is written as

$$|\Psi\rangle = |\psi_{\mathrm{A}}\rangle \otimes |\psi_{\mathrm{B}}\rangle \tag{19}$$

where

$$|\psi_{\mathrm{A}}\rangle = \int dx_{\mathrm{A}} \psi_{\mathrm{A}}(x_{\mathrm{A}})|x_{\mathrm{A}}\rangle, \tag{20}$$

and

$$|\psi_{\mathrm{B}}\rangle = \int dx_{\mathrm{B}} \psi_{\mathrm{B}}(x_{\mathrm{B}})|x_{\mathrm{B}}\rangle, \tag{21}$$

with $X_{\mathrm{A}}|x_{\mathrm{A}}\rangle = x_{\mathrm{A}}|x_{\mathrm{A}}\rangle$.

In equation (17), $|\mathbf{z}_l^0\rangle$ represents the spin coherent state at the lth site in the detector where a general spin coherent state is defined by [14,15]

$$|\mathbf{z}\rangle = |z_+\rangle \otimes |z_-\rangle \tag{22}$$

$$\equiv \left|\begin{pmatrix} z_+ \\ z_- \end{pmatrix}\right\rangle. \tag{23}$$

The spin coherent state is a simultaneous eigenstate of two annihilation operators b_+ and b_-,

$$b_\pm|\mathbf{z}\rangle = z_\pm|\mathbf{z}\rangle. \tag{24}$$

Using the annihilation and creation operators $b_\pm$ and $b_\pm^\dagger$, called Schwinger bosons [16], we can express the spin operator $\mathbf{S}$ in terms of them:

$$S^\pm = b_\pm^\dagger b_\mp, \tag{25}$$

and

$$S^z = \frac{1}{2}(N_+ - N_-), \tag{26}$$

with $N_\pm = b_\pm^\dagger b_\pm$ having a simultaneous eigenstate given by $|n_+, n_-\rangle = |n_+\rangle \otimes |n_-\rangle$.

Then, an eigenstate of S^z is found to be:

$$|S, m) \equiv |n_+ = S + m\rangle \otimes |n_- = S - m\rangle, \tag{27}$$

S being the magnitude of the spin $\mathbf{S}$ while m representing the eigenvalue of S^z.

Then the Bloch state (atomic coherent state) [17,18] is introduced as a superposition of $|S, m)$ or in an alternative form represented by

$$|S; \theta, \phi\rangle = \mathrm{e}^{-\mathrm{i}\phi S^z}\mathrm{e}^{-\mathrm{i}\theta S^y}|S, S). \tag{28}$$

These coherent states are related each other through a relation,

$$|\mathbf{z}_l^0\rangle = |z_{l,+}^0\rangle \otimes |z_{l,-}^0\rangle \tag{29}$$

$$= \mathrm{e}^{-|z_l^0|^2/2} \sum_{S=0}^{\infty} \frac{(z_l^0)^{2S}}{\sqrt{(2S)!}} |S;\theta_l^0,\phi_l^0\rangle \tag{30}$$

where

$$\mathbf{z}_l^0 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_l^0/2}\cos\frac{\theta_l^0}{2} \\ \mathrm{e}^{\mathrm{i}\phi_l^0/2}\sin\frac{\theta_l^0}{2} \end{pmatrix}. \tag{31}$$

Using equations (11) and (17), we can determine time evolution of the total density matrix exactly:

$$\begin{aligned} W(t) = \frac{1}{2}\int & dx_{\mathrm{A}}dx'_{\mathrm{A}}\psi_{\mathrm{A}}(x_{\mathrm{A}})\psi^*_{\mathrm{A}}(x'_{\mathrm{A}})|x_{\mathrm{A}}+v_{\mathrm{A}}t\rangle\langle x'_{\mathrm{A}}+v_{\mathrm{A}}t| \\ &\otimes \int dx_{\mathrm{B}}dx'_{\mathrm{B}}\psi_{\mathrm{B}}(x_{\mathrm{B}})\psi^*_{\mathrm{B}}(x'_{\mathrm{B}})|x_{\mathrm{B}}\rangle\langle x'_{\mathrm{B}}| \\ &\otimes\Bigg\{ (|+\rangle\langle+|)_{\mathrm{A}}\otimes(|-\rangle\langle-|)_{\mathrm{B}} \\ &\qquad\otimes\prod_{l=1}^{N}|\mathbf{z}_l^{0(int)}(x_{\mathrm{A}},t)\rangle\langle\mathbf{z}_l^{0(int)}(x'_{\mathrm{A}},t)| \\ &+(|-\rangle\langle-|)_{\mathrm{A}}\otimes(|+\rangle\langle+|)_{\mathrm{B}} \\ &\qquad\otimes\prod_{l=1}^{N}|\mathbf{z}_l^{0(non)}(t)\rangle\langle\mathbf{z}_l^{0(non)}(t)| \\ &-\mathrm{e}^{-\mathrm{i}\omega_I t}(|+\rangle\langle-|)_{\mathrm{A}}\otimes(|-\rangle\langle+|)_{\mathrm{B}} \\ &\qquad\otimes\prod_{l=1}^{N}|\mathbf{z}_l^{0(int)}(x_{\mathrm{A}},t)\rangle\langle\mathbf{z}_l^{0(non)}(t)| \\ &-\mathrm{e}^{\mathrm{i}\omega_I t}(|-\rangle\langle+|)_{\mathrm{A}}\otimes(|+\rangle\langle-|)_{\mathrm{B}} \\ &\qquad\otimes\prod_{l=1}^{N}|\mathbf{z}_l^{0(non)}(t)\rangle\langle\mathbf{z}_l^{0(int)}(x'_{\mathrm{A}},t)|\Bigg\} \end{aligned} \tag{32}$$

where

$$\mathbf{z}_l^{0(int)}(x_{\mathrm{A}},t) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega_l t}\Big(z_{l,+}^0\cos\frac{\Theta_l(x_{\mathrm{A}};t)}{2} \\ \qquad -\mathrm{i}z_{l,-}^0\mathrm{e}^{-\mathrm{i}\omega_l x_{\mathrm{A}}/v_{\mathrm{A}}}\sin\frac{\Theta_l(x_{\mathrm{A}};t)}{2}\Big) \\ \mathrm{e}^{\mathrm{i}\omega_l t}\Big(z_{l,-}^0\cos\frac{\Theta_l(x_{\mathrm{A}};t)}{2} \\ \qquad -\mathrm{i}z_{l,+}^0\mathrm{e}^{\mathrm{i}\omega_l x_{\mathrm{A}}/v_{\mathrm{A}}}\sin\frac{\Theta_l(x_{\mathrm{A}};t)}{2}\Big) \end{pmatrix}, \tag{33}$$

and

$$\mathbf{z}_l^{0(non)}(t) = \begin{pmatrix} z_{l,+}^0\mathrm{e}^{-\mathrm{i}\omega_l t} \\ z_{l,-}^0\mathrm{e}^{\mathrm{i}\omega_l t} \end{pmatrix}, \tag{34}$$

with $\omega_I \equiv \omega_{\mathrm{A}} - \omega_{\mathrm{B}}$.

As stated in the introduction, we are mainly interested in the spin dynamics and therefore, we eliminate the irrelevant variables of the orbital and the detector variables: This operation is written as $\mathrm{Tr}_{\Psi,\mathrm{D}}$ and thus we have

$$\rho_{\mathrm{AB}}(t) \equiv \mathrm{Tr}_{\Psi,\mathrm{D}}W(t) \tag{35}$$

$$\begin{aligned} &= \frac{1}{2}\Bigg\{(|+\rangle\langle+|)_{\mathrm{A}}\otimes(|-\rangle\langle-|)_{\mathrm{B}} \\ &\quad+(|-\rangle\langle-|)_{\mathrm{A}}\otimes(|+\rangle\langle+|)_{\mathrm{B}} \\ &\quad-\int dx_{\mathrm{A}}|\psi_{\mathrm{A}}(x_{\mathrm{A}})|^2\left[\mathrm{e}^{-\mathrm{i}\omega_I t}C(N;x_{\mathrm{A}},t)\right. \\ &\quad\left.\times(|+\rangle\langle-|)_{\mathrm{A}}\otimes(|-\rangle\langle+|)_{\mathrm{B}}+\mathrm{h.c.}\right]\Bigg\} \end{aligned} \tag{36}$$

where

$$\begin{aligned} C(N;x_{\mathrm{A}},t) = \prod_{l=1}^{N}\mathrm{e}^{-|z_l^0|^2}\exp\Bigg[|z_l^0|^2\Bigg(&\cos\frac{\Theta_l(x_{\mathrm{A}};t)}{2} \\ &-\mathrm{i}\sin\theta_l^0\cos(\phi_l^0-\omega_l x_{\mathrm{A}}/v_{\mathrm{A}})\sin\frac{\Theta_l(x_{\mathrm{A}};t)}{2}\Bigg)\Bigg]. \end{aligned} \tag{37}$$

We can further reduce the expressions (36) and (37) in the spin coherent state representation to the spin magnitude S space with the use of the formula [2],

$$\langle\mathbf{z}|\cdot|\mathbf{z}\rangle = \mathrm{e}^{-|z|^2}\sum_{S=0}^{\infty}\frac{|z|^{4S}}{(2S)!}\langle S;\theta,\phi|\cdot|S;\theta,\phi\rangle, \tag{38}$$

to obtain

$$\begin{aligned} \rho_{\mathrm{AB}}^{\mathrm{S}}(t) = \frac{1}{2}\Bigg\{&(|+\rangle\langle+|)_{\mathrm{A}}\otimes(|-\rangle\langle-|)_{\mathrm{B}} \\ &+(|-\rangle\langle-|)_{\mathrm{A}}\otimes(|+\rangle\langle+|)_{\mathrm{B}} \\ &-\int dx_{\mathrm{A}}|\psi_{\mathrm{A}}(x_{\mathrm{A}})|^2\left[\mathrm{e}^{-\mathrm{i}\omega_I t}C^{\mathrm{S}}(N;x_{\mathrm{A}},t)\right. \\ &\left.\times(|+\rangle\langle-|)_{\mathrm{A}}\otimes(|-\rangle\langle+|)_{\mathrm{B}}+\mathrm{h.c.}\right]\Bigg\} \end{aligned} \tag{39}$$

where

$$\begin{aligned} C^{\mathrm{S}}(N;x_{\mathrm{A}},t) = \prod_{l=1}^{N}\Bigg(&\cos\frac{\Theta_l(x_{\mathrm{A}};t)}{2} \\ &-\mathrm{i}\sin\theta_l^0\cos(\phi_l^0-\omega_l x_{\mathrm{A}}/v_{\mathrm{A}})\sin\frac{\Theta_l(x_{\mathrm{A}};t)}{2}\Bigg)^{2S_l}. \end{aligned} \tag{40}$$

Using these results and according to the method of reference [4], we find the von Neumann entropy $\mathcal{S}(t)$ of (3) with the definition $\lambda_1 \equiv \lambda_+$, $\lambda_2 \equiv \lambda_-$:

$$\mathcal{S}(t) = -(\lambda_+\ln\lambda_+ + \lambda_-\ln\lambda_-) \tag{41}$$

where

$$\lambda_\pm = \frac{1}{2}\pm\sqrt{|C|^2}, \tag{42}$$

$$C = -\frac{1}{2}e^{-\mathrm{i}\omega_I t}\int dx_{\mathrm{A}}|\psi_{\mathrm{A}}(x_{\mathrm{A}})|^2 C^{\mathrm{S}}(N;x_{\mathrm{A}},t). \tag{43}$$

These results will be used later.

2.2 Two detectors model

Next we generalize the previous situation to the case where both A and B move in opposite directions from the origin and not only A but also B interact with their respective detectors. Then the total Hamiltonian is written with obvious new notations:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{P}_+^{\rm A}\mathcal{H}_1^{\rm A} + \mathcal{P}_+^{\rm B}\mathcal{H}_1^{\rm B} \tag{44}$$

where

$$\mathcal{H}_0 = \mathcal{H}_{\rm S} + \mathcal{H}_{\rm D}, \tag{45}$$
$$\mathcal{H}_{\rm S} = v_{\rm A}P_{\rm A} + v_{\rm B}P_{\rm B} + \hbar\omega_{\rm A}I_{\rm A}^z + \hbar\omega_{\rm B}I_{\rm B}^z, \tag{46}$$
$$\mathcal{H}_{\rm D} = \sum_{l=1}^{N}\hbar\omega_l S_l^z + \sum_{l=-1}^{-N}\hbar\omega_l S_l^z, \tag{47}$$

and

$$\mathcal{H}_1^{\rm B} = \frac{1}{2}\sum_{l=-1}^{-N}\hbar\Omega_l(X_{\rm B}-x_l) \times \left(\mathrm{e}^{\mathrm{i}\omega_l X_{\rm B}/v_{\rm B}}S_l^- + \mathrm{e}^{-\mathrm{i}\omega_l X_{\rm B}/v_{\rm B}}S_l^+\right). \tag{48}$$

Time evolution operator is determined to give

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar} = {} & \mathcal{D}_{\rm A}^z(\{\omega_l X_{\rm A}/v_{\rm A}\})\mathcal{D}_{\rm B}^z(\{\omega_l X_{\rm B}/v_{\rm B}\})\mathrm{e}^{-\mathrm{i}\mathcal{H}'_0 t/\hbar} \\ & \times\left\{V_{\rm AB}(t)\mathcal{P}_+^{\rm A}\mathcal{P}_+^{\rm B} + V_{\rm A}(t)\mathcal{P}_+^{\rm A}\mathcal{P}_-^{\rm B}\right. \\ & \left. + V_{\rm B}(t)\mathcal{P}_-^{\rm A}\mathcal{P}_+^{\rm B} + \mathcal{P}_-^{\rm A}\mathcal{P}_-^{\rm B}\right\} \\ & \times \mathcal{D}_{\rm B}^z(\{\omega_l X_{\rm B}/v_{\rm B}\})^\dagger\mathcal{D}_{\rm A}^z(\{\omega_l X_{\rm A}/v_{\rm A}\})^\dagger\end{aligned} \tag{49}$$

where $V_{\rm B}(t)$ has a similar expression to equation (15) and

$$V_{\rm AB}(t) = V_{\rm A}(t)V_{\rm B}(t). \tag{50}$$

The initial density matrix for this system is of the form

$$W(0) = |I\rangle\langle I| \otimes |\Psi\rangle\langle\Psi| \otimes \prod_{l=1}^{N}|\mathbf{z}_l^0\rangle\langle\mathbf{z}_l^0| \otimes \prod_{l=-1}^{-N}|\mathbf{z}_l^0\rangle\langle\mathbf{z}_l^0|. \tag{51}$$

As in Section 2.1, we can get a reduced density matrix for the spin degree of freedom:

$$\begin{aligned}\rho_{\rm AB}(t) = \frac{1}{2}\Big\{ & (|+\rangle\langle+|)_{\rm A}\otimes(|-\rangle\langle-|)_{\rm B} \\ & + (|-\rangle\langle-|)_{\rm A}\otimes(|+\rangle\langle+|)_{\rm B} \\ & - \int dx_{\rm A}dx_{\rm B}|\psi_{\rm A}(x_{\rm A})|^2|\psi_{\rm B}(x_{\rm B})|^2 \\ & \times\left[e^{-i\omega_I t}C^{\rm A}(N;x_{\rm A},t)C^{{\rm B}*}(N;x_{\rm B},t)\right. \\ & \left.\times(|+\rangle\langle-|)_{\rm A}\otimes(|-\rangle\langle+|)_{\rm B} + h.c.\right]\Big\}\end{aligned} \tag{52}$$

where $C^{\rm A}(N;x_{\rm A},t) = C^S(N;x_{\rm A},t)$ and $C^{\rm B}(N;x_{\rm B},t)$ is the similar quantity with the replacement $\mathrm{A}\to\mathrm{B}$.

Thus we can obtain $\mathcal{S}(t)$ as before, but $\lambda_\pm$ replaced by

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{|C_{\rm AB}|^2}, \tag{53}$$

where

$$C_{\rm AB} = -\frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega_I t}\int dx_{\rm A}dx_{\rm B}|\psi_{\rm A}(x_{\rm A})|^2|\psi_{\rm B}(x_{\rm B})|^2 \times C^{\rm A}(N;x_{\rm A},t)C^{{\rm B}*}(N;x_{\rm B},t). \tag{54}$$

These will be used in Section 4.

3 Entanglement in boson detector model

In this section, we treat the BD model [3,8] referring only to differences from CH model and quote final results. In this model, the detectors are composed of harmonic oscillators:

$$\mathcal{H}_{\rm D} = \sum_{l=1}^{N}\hbar\omega_l a_l^\dagger a_l + \sum_{l=-1}^{-N}\hbar\omega_l a_l^\dagger a_l, \tag{55}$$

and

$$\mathcal{H}_1^{\rm A} = \frac{1}{2}\sum_{l=1}^{N}\hbar\Omega_l(X_{\rm A}-x_l) \times\left(\mathrm{e}^{\mathrm{i}\omega_l X_{\rm A}/v_{\rm A}}a_l + \mathrm{e}^{-\mathrm{i}\omega_l X_{\rm A}/v_{\rm A}}a_l^\dagger\right), \tag{56}$$

$$\mathcal{H}_1^{\rm B} = \frac{1}{2}\sum_{l=-1}^{-N}\hbar\Omega_l(X_{\rm B}-x_l) \times\left(\mathrm{e}^{\mathrm{i}\omega_l X_{\rm B}/v_{\rm B}}a_l + \mathrm{e}^{-\mathrm{i}\omega_l X_{\rm B}/v_{\rm B}}a_l^\dagger\right), \tag{57}$$

where a_l and $a_l^\dagger$ are the annihilation and the creation operators of the detector's harmonic oscillators, respectively.

Time evolution operator is written in the same form as equation (49). We have only to replace the operators $\mathcal{D}_{\rm A}^z(\{\phi_l\})$ by $\mathcal{D}_{\rm A}(\{\phi_l\})$,

$$\mathcal{D}_{\rm A}(\{\phi_l\}) = \prod_{l=1}^{N}\mathcal{D}_l(\phi_l), \tag{58}$$

$$\mathcal{D}_l(\phi_l) = \mathrm{e}^{-\mathrm{i}\phi_l a_l^\dagger a_l} \tag{59}$$

and a similar expression for $\mathcal{D}_{\rm B}^z(\{\phi_l\})$ and $V_{\rm A}(t)$, $V_{\rm B}(t)$ by

$$V_{\rm A}(t) = \exp\left[-\frac{\mathrm{i}}{2}\sum_{l=1}^{N}\Theta_l^{\rm A}(X_{\rm A};t)(a_l^\dagger + a_l)\right], \tag{60}$$

and

$$V_{\rm B}(t) = \exp\left[-\frac{\mathrm{i}}{2}\sum_{l=-1}^{-N}\Theta_l^{\rm B}(X_{\rm B};t)(a_l^\dagger + a_l)\right]. \tag{61}$$

Furthermore the initial state of the detectors is also replaced by usual boson coherent states:

$$W(0) = |I\rangle\langle I| \otimes |\Psi\rangle\langle\Psi| \otimes \prod_{l=1}^{N} |z_l^0\rangle\langle z_l^0| \otimes \prod_{l=-1}^{-N} |z_l^0\rangle\langle z_l^0|. \tag{62}$$

Finally, we have a reduced density matrix of the form:

$$\begin{aligned}\rho_{\mathrm{AB}}(t) = \frac{1}{2}\Big\{ & (|+\rangle\langle+|)_{\mathrm{A}} \otimes (|-\rangle\langle-|)_{\mathrm{B}} \\ & + (|-\rangle\langle-|)_{\mathrm{A}} \otimes (|+\rangle\langle+|)_{\mathrm{B}} \\ & - \int dx_{\mathrm{A}} dx_{\mathrm{B}} |\psi_{\mathrm{A}}(x_{\mathrm{A}})|^2 |\psi_{\mathrm{B}}(x_{\mathrm{B}})|^2 \\ & \times \left[\mathrm{e}^{-\mathrm{i}\omega_I t} C^{\mathrm{A}}(N; x_{\mathrm{A}}, t) C^{\mathrm{B}*}(N; x_{\mathrm{B}}, t) \right. \\ & \left. \times (|+\rangle\langle-|)_{\mathrm{A}} \otimes (|-\rangle\langle+|)_{\mathrm{B}} + h.c. \right] \Big\}\end{aligned} \tag{63}$$

where

$$\begin{aligned}C^{\mathrm{A}}(N; x_{\mathrm{A}}, t) = \prod_{l=1}^{N} \exp \Big[-\frac{\mathrm{i}}{2} \Big(& \mathrm{e}^{\mathrm{i}\omega_l x_{\mathrm{A}}/v_{\mathrm{A}}} z_l^0 \\ & + \mathrm{e}^{-\mathrm{i}\omega_l x_{\mathrm{A}}/v_{\mathrm{A}}} z_l^{0*} \Big) \Theta_l(x_{\mathrm{A}}; t) - \frac{1}{8} \Theta_l(x_{\mathrm{A}}; t)^2 \Big]\end{aligned} \tag{64}$$

$$\begin{aligned}C^{\mathrm{B}}(N; x_{\mathrm{B}}, t) = \prod_{l=-1}^{-N} \exp \Big[-\frac{\mathrm{i}}{2} \Big(& \mathrm{e}^{\mathrm{i}\omega_l x_{\mathrm{B}}/v_{\mathrm{B}}} z_l^0 \\ & + \mathrm{e}^{-\mathrm{i}\omega_l x_{\mathrm{B}}/v_{\mathrm{B}}} z_l^{0*} \Big) \Theta_l(x_{\mathrm{B}}; t) - \frac{1}{8} \Theta_l(x_{\mathrm{B}}; t)^2 \Big].\end{aligned} \tag{65}$$

The corresponding von Neumann entropy is also given by (3) with the following $\lambda_\pm$:

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{|C_{\mathrm{AB}}|^2}, \tag{66}$$

$$\begin{aligned}C_{\mathrm{AB}} = -\frac{1}{2} \mathrm{e}^{-\mathrm{i}\omega_I t} & \int dx_{\mathrm{A}} dx_{\mathrm{B}} |\psi_{\mathrm{A}}(x_{\mathrm{A}})|^2 |\psi_{\mathrm{B}}(x_{\mathrm{B}})|^2 \\ & \times C^{\mathrm{A}}(N; x_{\mathrm{A}}, t) C^{\mathrm{B}*}(N; x_{\mathrm{B}}, t).\end{aligned} \tag{67}$$

These results will be examined in the next section.

4 Numerical evaluation

In this section we will make explicit evaluation of the analytical results derived in the previous sections. In the following evaluation, we use a Gaussian form of $\Omega_l(x - x_l)$:

$$\Omega_l(x - x_l) = \frac{\Omega_l d}{\sqrt{2\pi\delta^2}} \mathrm{e}^{-(x-x_l)^2/2\delta^2}. \tag{68}$$

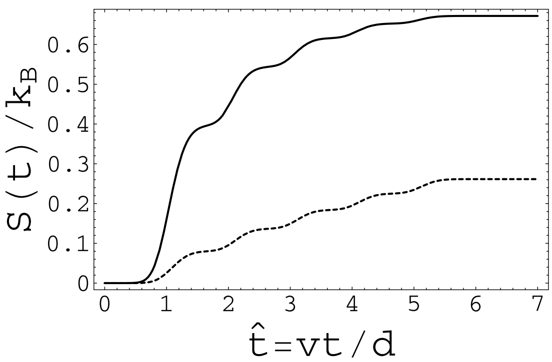

Fig. 1. Time evolution of $\mathcal{S}(t)/k_{\mathrm{B}}$ for a single detector CH model as a function of $\hat{t} = vt/d$ with $v_{\mathrm{A}} \equiv v$. The number of the detector spins $N = 5$. Other parameters are given by $\hat{\Omega}_l = \Omega_l d/v = 0.5$ and $\hat{\delta} = \delta/d = 0.25$. The solid line corresponds to $S_l = 5$ while the broken line to $S_l = 1/2$.

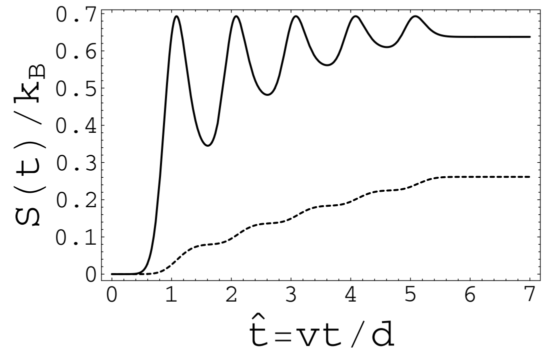

Fig. 2. The same figure as in Figure 1 for a single detector CH model keeping $N = 5$ and $S_l = 1/2$. The solid line for $\hat{\Omega}_l = 5$ and the broken line for $\hat{\Omega}_l = 0.5$.

For simplicity we set

$$|\psi_{\mathrm{A}}(x_{\mathrm{A}})|^2 = \delta(x_{\mathrm{A}}) \tag{69}$$

and

$$|\psi_{\mathrm{B}}(x_{\mathrm{B}})|^2 = \delta(x_{\mathrm{B}}). \tag{70}$$

And we put $v_{\mathrm{A}} = v$ and $v_{\mathrm{B}} = -v$.

In Figure 1 we show $\mathcal{S}(t)$ of equation (41) as a function of time by changing the magnitude of the detector spin for fixed value of N and the interaction strength which corresponds to relatively weak interaction. The entropy makes the stepwise increase when the incoming particle A passes each detector spin site. When the spin magnitude becomes large, we find faster decoherence with larger value of the entropy. Thus, the large value of the spin magnitude corresponds to a "classical measuring device".

In Figure 2 we show $\mathcal{S}(t)$ of equation (41) by changing the coupling strength. When the interaction becomes strong, there occurs frequent exchange effect between the particles and the detector resulting in the oscillating behavior.

In Figure 3 we plot the same quantity as a function of time with equation (54).

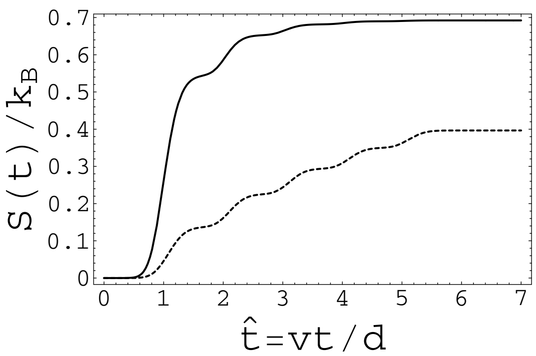

Fig. 3. The entropy $\mathcal{S}(t)/k_{\mathrm{B}}$ for the two detectors CH model as a function of time. The conditions are the same as in Figure 1.

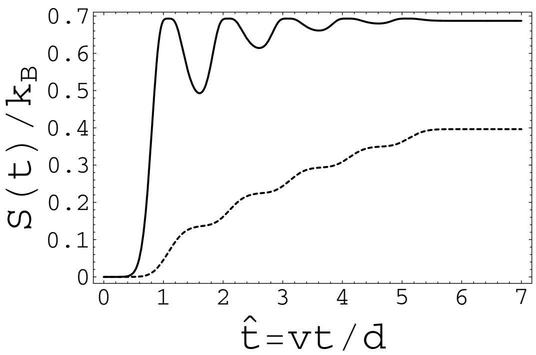

Fig. 4. The entropy $\mathcal{S}(t)/k_{\mathrm{B}}$ for the two detectors CH model as a function of time. The conditions are the same as in Figure 2.

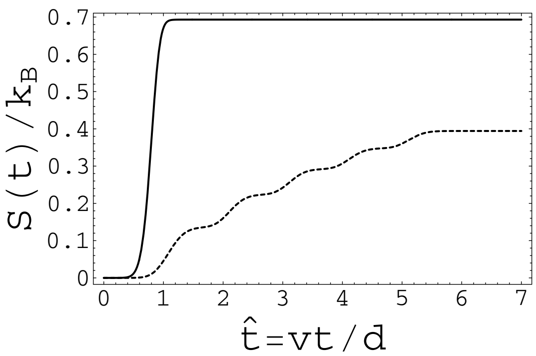

Fig. 5. Time evolution of $\mathcal{S}(t)/k_{\mathrm{B}}$ for the two detectors BD model with $N = 5$ and $z_l^0 = 0$. The solid line for $\hat{\Omega}_l = 5$ and the broken line for $\hat{\Omega}_l = 0.5$.

When we compare the behavior in Figure 3 with that in Figure 1, we find faster and larger increase in Figure 3. We used the same parameters in Figures 3 and 1, and therefore, this difference is due to the presence of the two detectors: the effect of the two detectors adds to increase the entropy of the composite A and B system to decohere the entangled state.

Figure 4 shows the same tendency as in Figure 3: Figure 4 should be compared with Figure 2. In Figure 5 we show a typical behavior of $\mathcal{S}(t)$ obtained from equation (66). The entropy increase is monotonous compared with CH model.

5 Discussions and conclusion

We studied decoherence phenomena on the basis of CH model and the BD model. We found that the initially entangled state suffers from the environmental fluctuations resulting in disappearance of the off-diagonal elements of the reduced density matrices, (39) and (52) for CH model and (63) for the BD model.

That is, the key quantity for CH model is $C^{\mathrm{S}}(N; x_{\mathrm{A}}, t)$ of (40):

$$C^{\mathrm{S}}(N; x_{\mathrm{A}}, t) = \prod_{l=1}^{N} c_l(S_l) \tag{71}$$

where

$$c_l(S_l) = \cos\frac{\Theta_l(x_{\mathrm{A}}; t)}{2} - \mathrm{i}\sin\theta_l^0 \cos(\phi_l^0 - \omega_l x_{\mathrm{A}}/v_{\mathrm{A}}) \sin\frac{\Theta_l(x_{\mathrm{A}}; t)}{2}. \tag{72}$$

From (72) we have

$$|c_l(S_l)| = \left(\cos^2\frac{\Theta_l(x_{\mathrm{A}}; t)}{2} + \sin^2\theta_l^0 \cos^2(\phi_l^0 - \omega_l x_{\mathrm{A}}/v_{\mathrm{A}}) \sin^2\frac{\Theta_l(x_{\mathrm{A}}; t)}{2}\right)^{S_l}. \tag{73}$$

We note that the quantity in the parenthesis of (73) is less than or equal to unity:

$$\cos^2\frac{\Theta_l(x_{\mathrm{A}}; t)}{2} + \sin^2\theta_l^0 \cos^2(\phi_l^0 - \omega_l x_{\mathrm{A}}/v_{\mathrm{A}}) \sin^2\frac{\Theta_l(x_{\mathrm{A}}; t)}{2} \le \cos^2\frac{\Theta_l(x_{\mathrm{A}}; t)}{2} + \sin^2\frac{\Theta_l(x_{\mathrm{A}}; t)}{2} = 1 \tag{74}$$

where equality holds under limited situations.

Thus we have

$$|c_l(S_l)| \longrightarrow 0 \tag{75}$$

and

$$C^{\mathrm{S}}(N; x_{\mathrm{A}}, t) \longrightarrow 0 \tag{76}$$

as $S_l \to \infty$.

We also note the following property of (71) due to (73) and (74):

$$C^{\mathrm{S}}(N; x_{\mathrm{A}}, t) \longrightarrow 0 \qquad (N \to \infty). \tag{77}$$

We have thus shown the vanishing of the off-diagonal elements of (39) and (52) when N and/or S_l become large.

For the BD model, the corresponding quantity which characterizes the decoherence property is given by (64):

$$C^{\mathrm{A}}(N : x_{\mathrm{A}}, t) = \prod_{l=1}^{N} c_l^{\mathrm{A}} \tag{78}$$

where

$$c_l^{\mathrm{A}} = \exp\left[-\frac{\mathrm{i}}{2}\left(\mathrm{e}^{\mathrm{i}\omega_l x_{\mathrm{A}}/v_{\mathrm{A}}} z_l^0 + \mathrm{e}^{-\mathrm{i}\omega_l x_{\mathrm{A}}/v_{\mathrm{A}}} z_l^{0*}\right)\Theta_l(x_{\mathrm{A}};t) - \frac{1}{8}\Theta_l(x_{\mathrm{A}};t)^2\right]. \quad (79)$$

From (79) we find that

$$|c_l^{\mathrm{A}}| = \exp\left[-\frac{1}{8}\Theta_l(x_{\mathrm{A}};t)^2\right] \leq 1. \quad (80)$$

The equality in (80) holds only when $t = 0$ and therefore, we have

$$C^{\mathrm{A}}(N; x_{\mathrm{A}}, t) \longrightarrow 0 \qquad (N \to \infty) \quad (81)$$

from (78).

In summary, we have shown vanishing of the off-diagonal elements of the reduced density matrices when N and/or S_l tend to infinity for CH model and $N \to \infty$ for the BD model: the initially entangled spin states are decohered in the *thermodynamic limit* $N \to \infty$ and/or in the *classical measuring apparatus limit* $S_l \to \infty$.

These decoherence quantities, $C^{\mathrm{S}}(N; x_{\mathrm{A}}, t)$, $C^{\mathrm{A}}(N; x_{\mathrm{A}}, t)$ and $C^{\mathrm{B}}(N; x_{\mathrm{B}}, t)$, play an important role also in von Neumann entropy $\mathcal{S}(t)$ through the eigenvalues $\lambda_{\pm}$ of (42), (53) and (66). As shown in the figures, decoherence of the entangled states reveals itself through the increasing tendency (sudden, oscillatory, and stepwise) of $\mathcal{S}(t)$. Strictly speaking, decoherence occurs only when the conditions $N \to \infty$ and/or $S_l \to \infty$ are satisfied. However, as seen from the figures, we found decoherence like phenomena even for finite values of N and S_l.

In conclusion, we have studied dynamical decoherence processes of the entangled states due to environmental fluctuations of the detector by exactly solving CH and the BD models. It is also explicitly shown that even when only one of the entangled particles is disturbed by the detector, the other state is largely influenced through the correlation between A and B. This is shown in the present work by obtaining the exact analytic results of the entropy and by explicit numerical evaluations. In our future work, we will examine other degree of entanglement [19,20] than the von Neumann entropy used in the present paper.

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